

# LOWER BOUNDS OF POTENTIAL BLOW-UP SOLUTIONS OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS IN $\dot{H}^{\frac{3}{2}}$

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ABSTRACT. We improve previous known lower bounds for Sobolev norms of potential blow-up solutions to the three-dimensional Navier-Stokes equations in  $\dot{H}^{3/2}$ .

## 1. INTRODUCTION

We consider the three-dimensional incompressible Navier-Stokes equations

$$\begin{aligned} (1.1) \quad & \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \\ & \nabla \cdot u = 0, \\ & u(x, 0) = u_0(x), \end{aligned}$$

where the velocity  $u(x, t)$  and the pressure  $p(x, t)$  are unknowns,  $\nu > 0$  is the kinematic viscosity coefficient, the initial data  $u_0(x) \in L^2(\Omega)$ , and the spatial domain  $\Omega$  may have periodic boundary conditions or  $\Omega = \mathbb{R}^3$ . The question of the regularity of solutions to (1.1) remains open and is one of the Clay Mathematics Institute Millennium Prize problems.

In 1934, Leray [7] published his formative work on the the fluid equations. He proved the existence of global weak solutions to (1.1) and proved that smooth solutions are unique in the class of Leray-Hopf solutions. He also showed that if  $\|u(t)\|_{H^1}$  is continuous on  $[0, T^*)$  and blows up at time  $T^*$ , then

$$(1.2) \quad \|u(t)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \frac{c}{(T^* - t)^{\frac{1}{4}}}.$$

Moreover, the bound for  $L^p$  norms for  $3 < p < \infty$ ,

$$(1.3) \quad \|u(t)\|_{L^p(\mathbb{R}^3)} \geq \frac{c_p}{(T^* - t)^{\frac{p-3}{2p}}},$$

have been known for a long time (see [7] and [6]). The Sobolev embedding  $\dot{H}^s(\mathbb{R}^3) \subset L^{\frac{6}{3-2s}}(\mathbb{R}^3)$  and (1.3) yield that

$$(1.4) \quad \|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c}{(T^* - t)^{\frac{2s-1}{4}}},$$

for  $\frac{1}{2} < s < \frac{3}{2}$  and  $\Omega = \mathbb{R}^3$ . Robinson, Sadowski, and Silva extended (1.4) in [10] for  $\frac{3}{2} < s < \frac{5}{2}$  for the whole space and in the presence of periodic boundary conditions. This bound is considered optimal for those values of  $s$ .

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When  $s > \frac{5}{2}$ , Benameur [1] showed

$$\|u(t)\|_{\dot{H}^s(\mathbb{R}^3)} \geq \frac{c(s)\|u(t)\|_{L^2(\mathbb{R}^3)}^{\frac{3-2s}{3}}}{(T^* - t)^{\frac{s}{3}}},$$

which was improved upon by Robinson, Sadowski, and Silva in [10] to

$$(1.5) \quad \|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c(s)\|u_0\|_{L^2(\Omega)}^{\frac{5-2s}{5}}}{(T^* - t)^{\frac{2s}{5}}},$$

when  $\Omega = \mathbb{T}^3$  or  $\Omega = \mathbb{R}^3$ .

The border cases  $s = \frac{3}{2}$  and  $s = \frac{5}{2}$  required separate treatment. For  $s = \frac{3}{2}$ , Robinson, Sadowski, and Silva had an epsilon correction for the case with periodic boundary conditions. In [5], Cortissoz, Montero, and Pinilla improved the bound for  $s = \frac{3}{2}$  on  $\mathbb{T}^3$ , but they had a logarithmic correction:

$$(1.6) \quad \|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \geq \frac{c}{\sqrt{(T^* - t)|\log(T^* - t)|}}.$$

For  $s = \frac{5}{2}$ , Cortissoz, Montero, and Pinilla [5] also found

$$(1.7) \quad \|u(t)\|_{\dot{H}^{\frac{5}{2}}(\Omega)} \geq \frac{c}{(T^* - t)|\log(T^* - t)|},$$

when  $\Omega = \mathbb{T}^3$  or  $\Omega = \mathbb{R}^3$ . In [8], the authors proved

$$(1.8) \quad \limsup_{t \rightarrow T^{*-}} (T^* - t) \|u(t)\|_{\dot{H}^{5/2}(\Omega)} \geq c.$$

In this paper, we improve the bound for the  $\dot{H}^{\frac{3}{2}}(\Omega)$ -norm to the optimal bound (1.4) when  $\Omega = \mathbb{R}^3$  or  $\Omega = \mathbb{T}^3$ . Our method is not contingent on rescaling arguments and thus works simultaneously for  $\mathbb{R}^3$  and  $\mathbb{T}^3$ . We stress the importance of the  $H^{3/2}$  norm, which scales to the  $L^\infty$  norm and corresponds to the uncovered limit of (1.4). We also note the significance of  $H^{5/2}$ , which is a critical space for the Euler equations and scales like  $B_{\infty,\infty}^1$ , the Beale-Kato-Majda space. Furthermore, the persistence of the logarithmic correction in estimate (1.7) is consistent with the recent result of Bourgain and Li [2] on the ill-posedness of the Euler equations in  $H^{5/2}$ .

*Remark 1.1.* The lower bound for the  $\dot{H}^{\frac{3}{2}}$ -norm of blow-up solutions was also presented in papers by Montero [9] and McCormick, Olson, Robinson, Rodrigo, Vidal-Lopez, and Zhou [8], which both appeared shortly after this paper.

Our methods differ from previous works as we utilize Littlewood-Paley decomposition of solutions  $u$  of (1.1) for much of the paper. We denote wave numbers as  $\lambda_q = 2^q$  (in some wave units). For  $\psi \in C^\infty(\Omega)$ , we define

$$\psi(\xi) = \begin{cases} 1 & : |\xi| \leq \frac{1}{2} \\ 0 & : |\xi| > 1. \end{cases}$$

Next define  $\phi(\xi) = \psi(\xi/\lambda_1) - \psi(\xi)$  and  $\phi_q(\xi) = \phi(\xi/\lambda_q)$ . Then

$$(1.9) \quad u = \sum_{q=-\infty}^{\infty} u_q,$$

in the sense of distributions, where the  $u_q$  is the  $q^{th}$  Littlewood-Paley piece of  $u$ . On  $\mathbb{R}^3$ , the Littlewood-Paley pieces are defined as

$$(1.10) \quad u_q(x) = \int_{\mathbb{R}^3} u(x-y) \mathcal{F}^{-1}(\phi_q)(y) dy,$$

where  $\mathcal{F}$  is the Fourier transform. In the periodic case, the Littlewood-Paley pieces are given by

$$(1.11) \quad u_q(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}(k) \phi_q(k) e^{ik \cdot x},$$

where (1.9) holds provided  $u$  has zero-mean. Moreover,  $u_q = 0$  in the periodic case when  $q < 0$ . We will use the notation

$$u_{\leq Q} = \sum_{q \leq Q} u_q, \quad u_{\geq Q} = \sum_{q \geq Q} u_q.$$

We define the homogeneous Sobolev norm of  $u$  as

$$(1.12) \quad \|u\|_{\dot{H}^s} = \left( \sum_{q=-\infty}^{\infty} \lambda_q^{2s} \|u_q\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Note that it corresponds to the nonhomogeneous Sobolev norm  $H^s$  in the periodic case.

We suppress  $L^p$  norm notation as  $\|\cdot\|_p := \|\cdot\|_{L^p}$ . We will also suppress the notation for domains for integrals and functional spaces, i.e.  $\int := \int_{\Omega}$ . All  $L^p$  and Sobolev spaces are over  $\Omega$ , where  $\Omega$  either has periodic boundary conditions or is the whole space  $\mathbb{R}^3$ , as described in the introduction (unless explicitly otherwise stated). The methods of proof apply to either domain. Sobolev spaces are denoted by  $H^s$  and homogeneous Sobolev spaces by  $\dot{H}^s$ . We will use the symbol  $\lesssim$  (or  $\gtrsim$ ) to denote that an inequality that holds up to an absolute constant.

## 2. BOUNDING BLOW-UP FOR $s = \frac{3}{2}$

We begin by testing the weak formulation of the Navier-Stokes equation with  $\lambda_q^{2s}(u_q)_q$  to obtain

$$(2.1) \quad \frac{d}{dt} (\lambda_q^{2s} \|u_q\|_2^2) = -\nu \lambda_q^{2s+2} \|u_q\|_2^2 + 2 \lambda_q^{2s} \int \text{Tr}[(u \otimes u)_q \cdot \nabla u_q] dx.$$

In the typical fashion, we write

$$(2.2) \quad (u \otimes u)_q = u_q \otimes u + u \otimes u_q + r_q(u, u),$$

for  $q > -1$ , where the remainder function is given by

$$(2.3) \quad r_q(u, u)(x) = \int \mathcal{F}^{-1}(\phi_q)(y) (u(x-y) - u(x)) \otimes (u(x-y) - u(x)) dy.$$

Thus, we rewrite the nonlinear term as

$$(2.4) \quad \int \text{Tr}[(u \otimes u)_q \cdot \nabla u_q] dx = \int r_q(u, u) \cdot \nabla u_q dx - \int u_q \cdot \nabla u_{\leq q+1} \cdot u_q dx.$$

**Lemma 2.1.** *The integral (2.4) corresponding to the nonlinear term in (2.1) is bounded above by*

$$\begin{aligned}
 \int \operatorname{Tr}[(u \otimes u)_q \cdot \nabla u_q] \, dx &\lesssim \lambda_q^{-1} \|u_q\|_2 \sum_{p=-\infty}^q \lambda_p^2 \|u_p\|_4^2 \\
 (2.5) \quad &+ \lambda_q \|u_q\|_2 \sum_{p=q+1}^{\infty} \|u_p\|_4^2 \\
 &+ \|u_q\|_2^2 \sum_{p=-\infty}^{q+1} \lambda_p^{\frac{5}{2}} \|u_p\|_2.
 \end{aligned}$$

*Proof.* We examine the two integrals on the right-hand side of (2.4) separately. By Hölder's inequality,

$$\int r_q(u, u) \cdot \nabla u_q \, dx \lesssim \|r_q(u, u)\|_2 \lambda_q \|u_q\|_2.$$

We use Littlewood-Paley decomposition and split the sum into low versus high modes to find

$$\begin{aligned}
 \|r_q(u, u)\|_2 &\lesssim \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\phi_q)(y)| \|u(x-y) - u(x)\|_4^2 \, dy \\
 &\lesssim \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\phi_q)(y)| \sum_{p=-\infty}^q \|(u(x-y) - u(x))_p\|_4^2 \, dy \\
 &\quad + \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\phi_q)(y)| \sum_{p=q+1}^{\infty} \|(u(x-y) - u(x))_p\|_4^2 \, dy.
 \end{aligned}$$

We apply the Mean-Value Theorem on the low modes and the triangle inequality on the high modes to arrive at

$$\begin{aligned}
 \|r_q(u, u)\|_2 &\lesssim \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\phi_q)(y)| |y|^2 \sum_{p=-\infty}^q \|\nabla u_p\|_4^2 \, dy \\
 &\quad + \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\phi_q)(y)| \sum_{p=q+1}^{\infty} \|u_p\|_4 \, dy \\
 &\lesssim \lambda_q^{-2} \sum_{p=-\infty}^q \lambda_p^2 \|u_p\|_4^2 + \sum_{p=q+1}^{\infty} \|u_p\|_4
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \int r_q(u, u) \cdot \nabla u_q \, dx &\lesssim \lambda_q^{-1} \|u_q\|_2 \sum_{p=-\infty}^q \lambda_p^2 \|u_p\|_4^2 \\
 (2.6) \quad &+ \lambda_q \|u_q\|_2 \sum_{p=q+1}^{\infty} \|u_p\|_4^2.
 \end{aligned}$$

For the second term of (2.4), we use a similar process as above in addition to Bernstein's inequality to find

$$(2.7) \quad \begin{aligned} \int u_q \cdot \nabla u_{\leq q+1} \cdot u_q \, dx &\lesssim \|u_q\|_2^2 \sum_{p=-\infty}^{q+1} \lambda_p \|u_p\|_\infty \\ &\lesssim \|u_q\|_2^2 \sum_{p=-\infty}^{q+1} \lambda_p^{5/2} \|u_p\|_2. \end{aligned}$$

Combining (2.6) and (2.7) yields the desired bound (2.5).  $\square$

Similar estimates were executed in [3] and [4]. We apply the bound obtained in Lemma 2.1 to write

$$(2.8) \quad \frac{d}{dt} \sum_{q=-\infty}^{\infty} (\lambda_q^{2s} \|u_q\|_2^2) \lesssim - \sum_{q=-\infty}^{\infty} (\nu \lambda_q^{2s+2} \|u_q\|_2^2) + 2(A + B + C),$$

where

$$(2.9) \quad A = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^q \lambda_q^{2s-1} \|u_q\|_2 \lambda_p^2 \|u_p\|_4^2,$$

$$(2.10) \quad B = \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^{2s+1} \|u_q\|_2 \|u_p\|_4^2,$$

$$(2.11) \quad C = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_q^{2s} \|u_q\|_2^2 \lambda_p^{5/2} \|u_p\|_2.$$

**Theorem 2.2.** *Let  $u$  be a solution to (1.1) with finite energy initial data. Then for  $s = \frac{3}{2}$ , the solution  $u$  satisfies the Riccati-type differential inequality*

$$(2.12) \quad \frac{d}{dt} \sum_{q=-\infty}^{\infty} (\lambda_q^3 \|u_q\|_2^2) \lesssim \sum_{q=-\infty}^{\infty} (\lambda_q^3 \|u_q\|_2^2)^2$$

*Proof.* We bound the nonlinear terms. First, we estimate (2.9) for  $s = \frac{3}{2}$ . We apply Bernstein's inequality in three-dimensions and we rewrite the sum

$$\begin{aligned} A &= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^q \lambda_q^2 \|u_q\|_2 \lambda_p^2 \|u_p\|_4^2 \\ &\lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^q \lambda_q^2 \|u_q\|_2 \lambda_p^{7/2} \|u_p\|_2^2 \\ &= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^q \lambda_{q-p}^{-1/2} (\lambda_q^{5/2} \|u_q\|_2) (\lambda_p^3 \|u_p\|_2^2). \end{aligned}$$

We apply the Cauchy-Schwartz inequality to yield

$$(2.13) \quad A \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^q \lambda_{q-p}^{-1/2} \left( \frac{\nu}{3} \lambda_q^5 \|u_q\|_2^2 \right) + \lambda_{q-p}^{-1/2} \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2.$$

Next we sum in  $p$  for the first term and exchange the order of summation and sum in  $q$  for the second term of (2.13):

$$(2.14) \quad A \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left( \lambda_q^5 \|u_q\|_2^2 \right).$$

To estimate (2.10) when  $s = \frac{3}{2}$ , first we apply Bernstein's inequality for three-dimensions to find

$$\begin{aligned} B &= \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^4 \|u_q\|_2 \|u_p\|_4^2 \\ &\lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^4 \|u_q\|_2 \lambda_p^{3/2} \|u_p\|_2^2. \end{aligned}$$

We rewrite the sum to look like

$$B \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} (\lambda_q^{3/2} \|u_q\|_2) (\lambda_p^{3/2} \|u_p\|_2) (\lambda_p^{5/2} \|u_p\|_2).$$

We apply Young's inequality with the exponents  $\theta_1 = \theta_2 = 4$  and  $\theta_3 = 2$  to yield

$$\begin{aligned} (2.15) \quad B &\lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 \\ &\quad + \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2 \\ &\quad + \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \left( \frac{\nu}{3} \lambda_p^5 \|u_p\|_2^2 \right). \end{aligned}$$

Next we sum in  $p$  for the first term and exchange the order of summation and sum in  $q$  for the second and third terms of (2.15). Note the summation in  $q$  converges:

$$B \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \sum_{p=-\infty}^{\infty} \left[ \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2 + \left( \frac{\nu}{3} \lambda_p^5 \|u_p\|_2^2 \right) \right].$$

Thus we arrive at the bound

$$(2.16) \quad B \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left( \lambda_q^5 \|u_q\|_2^2 \right).$$

Finally, we estimate (2.11) for  $s = \frac{3}{2}$ . We rewrite the sum

$$\begin{aligned} C &= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_q^3 \|u_q\|_2^2 \lambda_p^{5/2} \|u_p\|_2^2 \\ &= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_{q-p}^{-\delta} (\lambda_q^{3/2} \|u_q\|_2)^{2-\delta} (\lambda_q^{5/2} \|u_q\|_2^2)^{\delta} (\lambda_p^{3/2} \|u_p\|_2)^{\delta} (\lambda_p^{5/2} \|u_p\|_2^2)^{1-\delta}, \end{aligned}$$

where  $\delta$  is a small positive number we can choose. We apply Young's inequality with

$$\theta_1 = \frac{4}{2-\delta}, \quad \theta_2 = \frac{2}{\delta}, \quad \theta_3 = \frac{4}{\delta}, \quad \theta_4 = \frac{2}{1-\delta},$$

where we require  $\delta < 1$  to ensure the exponents are all positive and indeed  $\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} + \frac{1}{\theta_4} = 1$ . Then we have

$$(2.17) \quad \begin{aligned} C \lesssim & \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \left[ \lambda_{q-p}^{-\delta} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \lambda_{q-p}^{-\delta} \left( \frac{\nu}{6} \lambda_q^5 \|u_q\|_2^2 \right) \right] \\ & + \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \left[ \lambda_{q-p}^{-\delta} \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2 + \lambda_{q-p}^{-\delta} \left( \frac{\nu}{6} \lambda_p^5 \|u_p\|_2^2 \right) \right], \end{aligned}$$

For the first two terms of (2.17), we sum in  $p$ . For the third and fourth terms, we exchange the order of summation and sum in  $q$  to arrive at

$$\begin{aligned} C \lesssim & \sum_{q=-\infty}^{\infty} \left[ \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \left( \frac{\nu}{6} \lambda_q^5 \|u_q\|_2^2 \right) \right] \\ & + \sum_{p=-\infty}^{\infty} \left[ \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2 + \left( \frac{\nu}{6} \lambda_p^5 \|u_p\|_2^2 \right) \right]. \end{aligned}$$

Note  $\delta$  positive ensures the summation in  $q$  converges. Rewriting the above inequality yields

$$(2.18) \quad C \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left( \lambda_q^5 \|u_q\|_2^2 \right).$$

We use the estimates (2.14), (2.16), and (2.18) in (2.8) with  $s = \frac{3}{2}$  to get the Ricatti-type differential inequality

$$(2.19) \quad \frac{d}{dt} \sum_{q=-\infty}^{\infty} \left( \lambda_q^3 \|u_q\|_2^2 \right) \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2.$$

□

*Remark 2.3.* The method used to prove Theorem 2.2 works for  $\frac{1}{2} < s < \frac{5}{2}$ . Instead of (2.12), one must show

$$(2.20) \quad \frac{d}{dt} \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s} \|u_q\|_2^2 \right) \lesssim \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s-1}}.$$

In the proof for (2.20), one must treat the three cases  $\frac{1}{2} < s < \frac{3}{2}$ ,  $s = \frac{3}{2}$ , and  $\frac{3}{2} < s < \frac{5}{2}$  separately, but in analogous manners.

**Theorem 2.4.** *Let  $u$  be a smooth solution to (1.1) with finite energy initial data such that  $u$  loses regularity at time  $T^*$ . Then*

$$(2.21) \quad \|u(t)\|_{\dot{H}^{3/2}(\Omega)} \geq \frac{c}{\sqrt{T^* - t}},$$

for  $0 \leq t < T^*$  and  $\Omega = \mathbb{T}^3$  or  $\Omega = \mathbb{R}^3$ .

*Proof.* Let  $y(t) = \|u(t)\|_{\dot{H}^{3/2}}^2$ . By Theorem 2.2,  $y$  satisfies the differential inequality

$$(2.22) \quad \frac{d}{dt} y(t) \lesssim y(t)^2.$$

Rearranging the inequality and integrating from time  $t$  to blow-up time  $T^*$  yields

$$\int_{y(t)}^{\infty} \frac{dw}{w^2} \lesssim \int_t^{T^*} d\tau,$$

which becomes

$$\frac{1}{y(t)} \lesssim T^* - t.$$

Then, as desired

$$(2.23) \quad \|u(t)\|_{\dot{H}^{3/2}(\Omega)} \geq \frac{c}{\sqrt{T^* - t}},$$

for  $0 \leq t < T^*$  and  $\Omega = \mathbb{T}^3$  or  $\Omega = \mathbb{R}^3$ .  $\square$

*Remark 2.5.* The procedure in Theorem 2.4 can be applied to (2.20) for  $y(t) = \|u(t)\|_{\dot{H}^s(\Omega)}^2$  to yield

$$(2.24) \quad \|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c}{(T^* - t)^{\frac{2s-1}{4}}},$$

for  $\frac{1}{2} < s < \frac{5}{2}$ ,  $0 \leq t < T^*$ , and  $\Omega = \mathbb{R}^3$  or  $\Omega = \mathbb{T}^3$ .

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